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EXTENSIONS OF CHEVALLEY GROUPS

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ABSTRACT

Let G_0 be a split simple Chevalley group of any type over the field K and G its universal group; and let $\hat{G}_0$ be the group of automorphisms of the corresponding Chevalley algebra, L_K, generated by G_0 and all the diagonal automorphisms. A group $\hat{G}$ (and appropriate homorphisms) is constructed which generalizes the group $GL_n(K)$ when G_0 is specialized to type A_{n-1}.

Introduction

If K is any field, K^* its non-zero elements, and if n is any positive integer then the following commutative diagram, in which all rows and columns are exact, is well known when $G = SL_n(K)$, $G_0 = PSL_n(K)$, $\hat{G} = GL_n(K)$, $\hat{G}_0 = PGL_n(K)$, Z is the center of $SL_n(K)$, and the homomorphisms are the obvious ones:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z & \xrightarrow{\rho} & K^* & \xrightarrow{(\cdot)^n} & K^{*n} & \longrightarrow & 1 \\
 & & \downarrow & & \delta\downarrow & & \downarrow & & \\
1 & \longrightarrow & G & \xrightarrow{i} & \hat{G} & \xrightarrow{\det} & K^* & \longrightarrow & 1 \\
 & & \pi\downarrow & & \mathrm{res}\downarrow & & \downarrow\mathrm{nat} & & \\
1 & \longrightarrow & G_0 & \longrightarrow & \hat{G}_0 & \xrightarrow{d} & K^*/K^{*n} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{1}
$$

Here, as in the rest of the paper, we use the notation of Dieudonné [3] for the classical groups. If the three groups $SL_n(K)$, $PSL_n(K)$, and $PGL_n(K)$ are

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viewed as Chevalley groups of type A_{n-1} over K, then it is reasonable to view this diagram in the wider context of all Chevalley groups. Thus, $SL_n(K)$ and $PSL_n(K)$ are special cases of the universal and adjoint Chevalley groups, G and G_0; while $PGL_n(K)$ is the group $\hat{G}_0$, which is generated by G_0 and all diagonal automorphisms of the underlying Chevalley algebra, L_K, over K. ($\hat{G}_0$ is the group $\hat{G}$ defined by Carter in [2, p. 118], and the group G defined by Seligman in [4, p. 51].) We then wish to construct a group $\hat{G}$, and appropriate mappings, so that we have the commutative diagram (1) with exact rows and columns. Here n will be the order of the quotient of the weight lattice by the root lattice.

Our construction is in fact a two step affair in which, beginning with a root system Δ of rank l, we first construct a torus $\hat{H}$ of rank $l+1$ for which a corresponding diagram [see diagram (2)] holds with G, G_0, and $\hat{G}_0$ replaced by suitable maximal split torii within them. The group $\hat{G}$ then appears as a quotient of a semi-direct product of G and $\hat{H}$.

As long as Δ is not of type D_l with l even, the quotient of the weight lattice L_1 by the root lattice L_0 is ciclic. Our construction requires the choice of an element of L_1 which generates L_1/L_0. In the case of D_l, l even, L_1/L_0 is the direct product of two cyclic groups of order two, and the construction is modified accordingly with two generators being chosen. We are able to show that the outcome of the construction is essentially independent of the choice of generator except possibly in the case when Δ is of type A_l. Of particular interest in the construction is the precise description of det: $\hat{G} \to K^*$, which when viewed abstractly, is nothing more than a certain evaluation map.

Realizations of the universal and adjoint groups G and G_0 are well known, see [5]; while the groups $\hat{G}_0$ are identified in [4]. The question of identification of the groups $\hat{G}$ is rather interesting, but not as yet totally settled. When Δ is of type A_l and we choose the highest weight of the standard representation as the generator of L_1/L_0 then $\hat{G}$ is, as expected, $GL_{l+1}(K)$ and det is the usual determinant mapping. For Δ of type B_l, $\hat{G}$ is the group of invertible elements in the even Clifford algebra, $C^+(V)$, of an orthogonal geometry V with maximal Witt index, which normalize V; and det is the inverse of the norm map of $C^+(V)$ restricted to $\hat{G}$. When Δ is of type C_l, $\hat{G}$ is the group of similitudes of a symplectic geometry and det is the multiplier map. When Δ is of type D_l the identification of $\hat{G}$ is not clear to us, but we note that for l odd, the realization of $\hat{G}$ and det will involve groups related to orthogonal geometries, being tied up with fourth powers in K^*, and hence seems rather unusual.

Our main theorem is the following:

THEOREM. *Let K be a field and Δ a root system of rank l. Let L_1 and L_0 be the weight and root lattices of Δ and let $n = |L_1/L_0|$. Let L_K be the Chevalley algebra of Δ over K and let G and G_0 be the corresponding universal and adjoint Chevalley groups. Let $\hat{G}_0$ be the group in* Aut(L_K) *generated by G_0 and the diagonal automorphisms.*

(1) *If L_1/L_0 is cyclic, then for each choice of a $\Lambda \in L_1$ which generates L_1/L_0 there is a group $\hat{G}$ and mappings ρ, δ, i,* det, res, *for which the commutative diagram* (1) *holds with exact rows and columns.*

(2) *If L_1/L_0 is the Klein four-group then for each pair Λ^+, Λ^- of elements of L_1 generating L_1/L_0 there is a group $\hat{G}$ and mappings ρ, δ, i* det, res, *for which the commutative diagram* (1′) *holds with exact rows and columns.*

(3) *When Δ is not of type A_l the group $\hat{G}$ is, up to isomorphism compatible with the diagram, independent of the choice of generators.*

Notation is introduced in Section 1. In Section 2 we construct the toroidal diagram (2) when L_1/L_0 is cyclic, and in Section 3 we complete this to the full diagram (1). The case of non-cyclic L_1/L_0 is dealt with in Section 4. In Section 5 we discuss the uniqueness of $\hat{G}$.

It is perhaps worthwhile to remark that the existence of our diagrams, and some usual properties of Chevalley groups, easily imply that any subgroup of $\hat{G}$ which is normalized by elements from G, either contains G or lies in the center of $\hat{G}$.

1. Notation and conventions

Let Δ be finite indecomposable root system determined by a Cartan matrix (A_{ij}). Let $\alpha_1, \cdots, \alpha_l$ be a fundamental system of roots of Δ, so Δ is a subset of the root lattice $L_0 = Z\alpha_1 \oplus \cdots \oplus Z\alpha_l$. Let $A = Zh_1 \oplus \cdots \oplus Zh_l$ be a free abelian group on l generators $h_1, \cdots, h_l$. Each $\beta = \Sigma_i z_i \alpha_i \in L_0$ determines a homomorphism of A into Z by $\beta(h_j) = \Sigma_i z_i A_{ij}$. Put $L_1 = \mathrm{Hom}(A, Z)$, the lattice of weights. The above mapping of L_0 to L_1 is an injective homomorphism by which we consider L_0 as a subgroup of L_1. The quotient group L_1/L_0 is finite and is given by Table I (see [5]).

Table I

Type of Δ	A_l	B_l	C_l	D_{2m+1}	D_{2m}	E_6	E_7	E_8, F_4, G_2
L_1/L_0	$Z/(l+1)Z$	$Z/2Z$	$Z/2Z$	$Z/4Z$	$Z/2Z \times Z/2Z$	$Z/3Z$	$Z/2Z$	$\{1\}$

For $\beta \in L_1$, $h \in A$, we will usually write $\langle \beta, h \rangle$ for $\beta(h)$.

Let K be a field and K^* its non-zero elements. For each subgroup L lying between L_0 and L_1 there is a Chevalley group G_L over K. We use the notation of [5] for the various objects associated with this group. Thus, G_L has an abelian subgroup H_L isomorphic to the direct product of l copies of K^*, and a set of one-parameter groups X_α, $\alpha \in \Delta$, each of which is normalized by H_L. More precisely each $h \in H_L$ can be written in the form $h_1(t_1)\cdots h_l(t_l)$ where each $t_i \in K^*$ and the h_i satisfy $h_i(t)h_i(t') = h_i(tt')$. Then, for $x_\alpha(s) \in X_\alpha$ we have

$$\prod_{i=1}^{l} h_i(t_i)x_\alpha(s)\left(\prod_{i=1}^{l} h_i(t_i)\right)^{-1} = x_\alpha\left(s\prod_{i=1}^{l} t_i^{\langle \alpha, h_i\rangle}\right).$$

Moreover, $\Pi_{i=1}^{l} h_i(t_i) = 1$ if and only if $\Pi_{i=1}^{l} t_i^{\langle \mu, h_i\rangle} = 1$ for all $\mu \in L$. Each $h = \Pi_{i=1}^{l} h_i(t_i) \in H_L$ determines an element of $\mathrm{Hom}(L, K^*)$ by $\mu \to \Pi_{i=1}^{l} t_i^{\langle \mu, h_i\rangle}$, for all $\mu \in L$. The resulting mapping of H_L to $\mathrm{Hom}(L, K^*)$ is an injective homomorphism via which we identify H_L with a subgroup of $\mathrm{Hom}(L, K^*)$.

In the sequel we will be interested in only the two extreme cases of L, that is L_0 and L_1. We will denote G_{L_0} and H_{L_0} by G_0 and H_0 respectively, and G_{L_1} and H_{L_1} by G and H respectively. We note that $H = \mathrm{Hom}(L_1, K^*)$.

The center Z of G lies in H and consists of elements $\Pi_{i=1}^{l} h_i(t_i)$ for which $\Pi_{i=1}^{l} t_i^{\langle \mu, h_i\rangle} = 1$ for all $\mu \in L_0$. Z can be identified with $\mathrm{Hom}(L_1/L_0, K^*)$, and we will make this identification. The group G_0 is a natural homomorphic image of G by a homomorphism π with kernel Z. We also note that H_0 is the subgroup of elements of $\mathrm{Hom}(L_0, K^*)$ which can be extended to elements of $H = \mathrm{Hom}(L_1, K^*)$.

Let $\hat{H}_0 = \mathrm{Hom}(L_0, K^*)$. Then if L_K denotes the Chevalley algebra associated to K and Δ, we have that both $\hat{H}_0$ and G_0 are subgroups of $\mathrm{Aut}(L_K)$. $\hat{G}_0$ is defined to be the subgroup of $\mathrm{Aut}(L_K)$ generated by G_0 and $\hat{H}_0$. Each $h \in \hat{H}_0$ determines the automorphism of L_K whose effect on the root space L_α, $\alpha \in \Delta$, is multiplication by the scalar $h(\alpha)$. G_0 is normalized by $\hat{H}_0$, $\hat{H}_0 \cap G_0 = H_0$, and for $\alpha \in \Delta$, $h \in \hat{H}_0$, $t \in K^*$, we have $hx_\alpha(t)h^{-1} = x_\alpha(h(\alpha)t)$.

2. Construction of $\hat{H}$ when L_1/L_0 is cyclic

Thoughout this section and the next we assume that L_1/L_0 is cyclic of order n. We are going to construct a group $\hat{H}$ and mappings i, ρ, det, δ, d, and res, so that the following diagram is commutative and has exact rows and columns.

Here all unmarked non-trivial mappings are identity mappings, nat : $K^* \to K^*/K^{*n}$ is the natural homomorphism and $(\cdot)^n : K^* \to K^{*n}$ is the nth power homomorphism.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z & \xrightarrow{\rho} & K^* & \xrightarrow{(\cdot)^n} & K^{*n} & \longrightarrow & 1 \\
 & & \downarrow & & \delta\downarrow & & \downarrow & & \\
1 & \longrightarrow & H & \xrightarrow{i} & \hat{H} & \xrightarrow{\det} & K^* & \longrightarrow & 1 \\
 & & \pi\downarrow & & \mathrm{res}\downarrow & & \downarrow\mathrm{nat} & & \\
1 & \longrightarrow & H_0 & \longrightarrow & \hat{H}_0 & \xrightarrow{d} & K^*/K^{*n} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{2}
$$

Since L_1/L_0 is cyclic we can choose a weight Λ in L_1 such that $\Lambda + L_0$ generates L_1/L_0. Then $L_1 = \mathbf{Z}\Lambda + \mathbf{Z}\alpha_1 + \cdots + \mathbf{Z}\alpha_l$. Embed L_0 in a free group $\hat{L} = \mathbf{Z}\alpha_0 \oplus \cdots \oplus \mathbf{Z}\alpha_l$ of rank $l+1$. The group $\hat{H}$ is defined to be $\mathrm{Hom}(\hat{L}, K^*)$.

(a) Definition of ρ. Let $f \in Z = \mathrm{Hom}(L_1/L_0, K^*)$. We then set $\rho(f) = f(\Lambda + L_0)$. Clearly $\rho(f)$ is an nth root of 1 in K^* and ρ is a group homomorphism making the top row of the diagram exact.

(b) Definition of res and δ. The map res: $\hat{H} \to \hat{H}_0$ just sends the function $f \in \hat{H}$ to its restriction to L_0. Thus, res is surjective and its kernel the group of $f \in \hat{H}$ for which $f(\alpha_i) = 1$, $1 \le i \le l$. For each $c \in K^*$ define $f_c \in \hat{H}$ by $f_c(\alpha_0) = c^{-1}$, $f_c(\alpha_i) = 1$, $1 \le i \le l$. The map δ is given by $\delta(c) = f_c$, for all $c \in K^*$, and it is immediate that the center column is exact.

(c) Definition of i and det. The homomorphism i is defined as follows: $i(f)(\alpha_0) = f(\Lambda)^{-1}$, $i(f)(\alpha_j) = f(\alpha_j)$, $1 \le j \le l$.

We know that $n\Lambda \in L_0$, thus, $n\Lambda = z_1\alpha_1 + \cdots + z_l\alpha_l$ for some $z_i \in \mathbf{Z}$, and g.c.d. $(n, z_1, \cdots, z_l) = 1$. Let $z_0 = n$. We then define the homomorphism $\det : \hat{H} \to K^*$ by $\det(f) = f(w)$ for all $f \in H$, where $w = -(z_0\alpha_0 + \cdots + z_l\alpha_l)$. With these definitions the center row is exact.

The commutativity of each of the squares other than the lower right one is easy to check. The definition of $d: \hat{H}_0 \to K^*/K^{*n}$ is forced by the rest of the diagram; namely for $f \in \hat{H}_0$ and $\hat{f}$ any preimage of f in $\hat{H}$, $d(f) = \mathrm{nat}(\det(\hat{f}))$.

3. Construction of $\hat{G}$ when L_1/L_0 is cyclic

Since G_0 is normalized by $\hat{H}_0$, there is a homomorphism $\psi_0: \hat{H}_0 \to \mathrm{Aut}(G_0)$ given by $h \in \hat{H}_0 \mapsto (g \mapsto hgh^{-1})$ for all $g \in G_0$. As noted in Section 1 we have that $\psi_0(h)x_\alpha(t) = x_\alpha(h(\alpha)t)$ for all $\alpha \in \Delta$, $t \in K$.

We can define a similar homomorphism $\psi : \hat{H} \to \mathrm{Aut}(G)$ with the property that $\psi(h)x_\alpha(t) = x_\alpha(h(\alpha)t)$ for all $h \in \hat{H}$, $\alpha \in \Delta$, and $t \in K$. Indeed, G is known to be defined by the generators $x_\alpha(t)$, $\alpha \in \Delta$, $t \in K$, and the relations given in [5, sec. 6]. For each $h \in \hat{H}$, the mapping $\psi(h)$ taking $x_\alpha(t)$ into $x_\alpha(h(\alpha)t)$ preserves these relations and hence determines an automorphism of G.

Let $\bar{\pi} : \mathrm{Aut}(G) \to \mathrm{Aut}(G_0)$ be defined as follows: $\bar{\pi}(\lambda)\pi(g) = \pi(\lambda(g))$, for all $\lambda \in \mathrm{Aut}(G)$, $g \in G$. We then have the following commutative diagram:

$$(3)\qquad \begin{array}{ccc} \hat{H} & \xrightarrow{\psi} & \mathrm{Aut}(G) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ \hat{H}_0 & \xrightarrow[\psi_0]{} & \mathrm{Aut}(G_0) \end{array}$$

Let $\mu = i^{-1} : i(H) \to H$. If $h \in H$, then for $x_\alpha(t) \in G$ we have $hx_\alpha(t)h^{-1} = x_\alpha(h(\alpha)t) = \psi(i(h))x_\alpha(t)$. Thus, for all $\hat{h} \in i(H)$ and for all $g \in G$, $\psi(\hat{h})g = \mu(\hat{h})g\mu(\hat{h})^{-1}$.

Let $G \times_\psi \hat{H}$ be the semi-direct product of G and $\hat{H}$ relative to ψ. Thus $G \times_\psi \hat{H} = \{(g, h) | g \in G, h \in \hat{H}\}$ with multiplication $(g_1, h_1)(g_2, h_2) = (g_1 \psi(h_1) g_2, h_1 h_2)$. Our group $\hat{G}$ is $(G \times_\psi \hat{H})/M$, where $M = \{(g, h) \in G \times_\psi \hat{H} \mid \det(h) = 1 \text{ and } \mu(h) = g^{-1}\}$. Some simple computations show that M is a normal central subgroup of $G \times_\psi \hat{H}$.

For notation we denote the image of the element $(g, h) \in G \times_\psi \hat{H}$ in $\hat{G}$, under the natural homomorphism, by $[g, h]$.

Now we will define mappings det, i, res, d, δ which make diagram (1) commutative with exact rows and columns. Each of these mappings will extend the corresponding ones of diagram (2).

(a) Definition of i and det. The mapping i is the composite $g \mapsto (g, 1) \mapsto [g, 1]$ and is injective. The map $\det : \hat{G} \to K^*$ is defined by $\det[g, h] = \det(h)$. This makes the center row exact.

(b) Definition of res and δ. Define $R : G \times_\psi \hat{H} \to \hat{G}_0$ by $R(g, h) = \pi(g)\,\mathrm{res}(h)$. Since π and res are surjective and $\hat{G}_0 = G_0\hat{H}_0$, R is a surjective homomorphism.

If $(g, h) \in M$, then $\det(h) = 1$ and $\mu(h) = g^{-1}$. Thus, $\pi(\mu(h)) = \pi(g)^{-1}$. But from (2), $\pi(\mu(h)) = \mathrm{res}(h)$, and hence $R(g, h) = \pi(g)\,\mathrm{res}(h) = 1$. Thus, R can be factored through M, to obtain our map $\mathrm{res} : \hat{G} \to \hat{G}_0$, which is surjective.

Define $\delta : K^* \to \hat{G}$ by $\delta(c) = [1, \delta(c)]$. δ is injective and $\delta(K^*)$ is contained in the kernel of $\mathrm{res} : \hat{G} \to \hat{G}_0$. Conversely, suppose that $\mathrm{res}[g, h] = 1$. From

$\pi(g)\operatorname{res}(h)=1$, we obtain $\operatorname{res}(h)=\pi(g^{-1})\in\pi(G)=G_0$, and so $\operatorname{res}(h)\in\hat{H}_0\cap G_0=H_0$. It follows that there is an $h_1\in\hat{H}$ such that $\pi\mu(h_1)=\operatorname{res}(h)$. Then $\operatorname{res}(h_1)=\operatorname{res}(h)$, so $h_1^{-1}h=\delta(k)$ for some $k\in K^*$. Once we prove that $[g,h_1]\in\delta(K^*)$, then we will have $[g,h_1][1,\delta(k)]=[g,h_1\delta(k)]=[g,h]\in\delta(K^*)$, and we are done.

Consider $[g,h_1]$. We have $\operatorname{res}[g,h_1]=\pi(g)\operatorname{res}(h_1)=\pi(g)\operatorname{res}(h)=1$. Now $\operatorname{res}(h_1)=\pi\mu(h_1)$, so $\pi(g\mu(h_1))=1$, whence $g\mu(h_1)=z\in Z$. We will show that $[1,\delta\rho(z)]=[g,h_1]$, and so $[g,h_1]=\delta([1,\rho(z)])\in\delta(K^*)$. Now $[1,\delta\rho(z)]=[g,h_1]$ if and only if $(g,\delta(\rho(z))^{-1}h_1)\in M$, which in turn is true if and only if $\det(\delta(\rho(z))^{-1}h_1)=1$ and $g^{-1}=\mu(\delta(\rho(z))^{-1}h_1)$. Since $\delta(\rho(z))=i(z)$, we have $\det(\delta(\rho(z))^{-1}=1$, and since $h_1\in i(H)$, $\det(\delta(\rho(z))^{-1}h_1)=1$. Finally, $\mu(\delta(\rho(z))^{-1}h_1)=\mu(i(z)^{-1})\mu(h_1)=z^{-1}\mu(h_1)=g^{-1}$. This completes the proof of the exactness of the center column.

The commutativity of all squares of (1), other than the lower right hand one, is an easy matter to verify. Once again the definition of d is forced in order to make the lower right hand square commute.

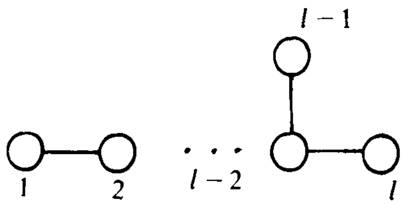
4. Construction of $\hat{H}$ and $\hat{G}$ when $L_1/L_0=Z/2Z\times Z/2Z$

The situation $L_1/L_0\cong Z/2Z\times Z/2Z$ occurs if and only if Δ is of type D_l, with l even. In this case every occurrence of K^* in diagram (3) is replaced by $K^*\times K^*$. Thus, we will construct $\hat{G}$ and appropriate mappings so that we have the following commutative diagram with exact rows and columns.

$$
\text{(1')}\qquad
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z & \xrightarrow{\rho} & K^*\times K^* & \xrightarrow{(\cdot)^2\times(\cdot)^2} & K^{*2}\times K^{*2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow & & \\
1 & \longrightarrow & G & \xrightarrow{i} & \hat{G} & \xrightarrow{\det} & K^*\times K^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\mathrm{res}} & & \downarrow{\scriptstyle\mathrm{nat}} & & \\
1 & \longrightarrow & G_0 & \longrightarrow & \hat{G}_0 & \xrightarrow{d} & K^*/K^{*2}\times K^*/K^{*2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

As before, we begin with the construction of a group $\hat{H}$ for which (1') holds when every occurrence of "G" is replaced by the corresponding "H" group.

We will assume that the fundamental roots $\alpha_1, \cdots, \alpha_l$ of Δ are labelled compatibly with the Dynkin diagram



Then the weights

$$\lambda^{+} = \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + \left(\frac{l}{2}\right)\alpha_{l-1} + \left(\frac{l-2}{2}\right)\alpha_l\right)$$

and

$$\lambda^{-} = \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + \left(\frac{l-2}{2}\right)\alpha_{l-1} + \left(\frac{l}{2}\right)\alpha_l\right)$$

determine a pair of generators for L_1/L_0, see [1]. We let $\hat{L} = Z\alpha^{+} \oplus Z\alpha^{-} \oplus Z\alpha_1 \oplus \cdots \oplus Z\alpha_l$, be a free abelian group on $l+2$ generators, and put $\hat{H} = \mathrm{Hom}(\hat{L}, K^*)$. Let

$$w^{+} = -\left(2\alpha^{+} + \alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + \left(\frac{l}{2}\right)\alpha_{l-1} + \left(\frac{l-2}{2}\right)\alpha_l\right)$$

and

$$w^{-} = -\left(2\alpha^{-} + \alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + \left(\frac{l-2}{2}\right)\alpha_{l-1} + \left(\frac{l}{2}\right)\alpha_l\right).$$

The map $i : H \to \hat{H}$ is defined as follows: if $f \in H$, we let $i(f)(\alpha_j) = f(\alpha_j)$ for $1 \leq j \leq l$, while $i(f)(\alpha^{+}) = f(\lambda^{+})^{-1}$ and $i(f)(\alpha^{-}) = f(\lambda^{-})^{-1}$. The map $\det : \hat{H} \to K^* \times K^*$ is defined by $\det(f) = (f(w^{+}), f(w^{-}))$, for all $f \in \hat{H}$. The exactness of $1 \to H \to \hat{H} \to K^* \times K^* \to 1$ follows from the fact that the homomorphism $\phi : \hat{L} \to L$ defined by $\alpha_j \mapsto \alpha_j$, $1 \leqq j \leqq l$, $\alpha^{+} \mapsto -\lambda^{+}$, $\alpha^{-} \mapsto -\lambda^{-}$, has kernel $Zw^{+} + Zw^{-}$.

The mapping $\delta : K^* \times K^* \to \hat{H}$ is defined by $(c^{+}, c^{-}) \mapsto f$, where $f(\alpha_j) = 1$ for $1 \leqq j \leqq l$, $f(\alpha^{+}) = (c^{+})^{-1}$, $f(\alpha^{-}) = (c^{-})^{-1}$, for all $(c^{+}, c^{-}) \in K^* \times K^*$. $\rho : z \to K^* \times K^*$ is $f \mapsto (f(\lambda^{+}), f(\lambda^{-}))$, for all $f \in Z = \mathrm{Hom}(L_1/L_0, K^*)$. Finally, $\mathrm{res} : \hat{H} \to \hat{H}_0$ is simply restriction of the functions in $\hat{H}$ to the domain L_0.

$\hat{G}$ is now easily constructed in precisely the same manner as given in Section 3.

5. Uniqueness questions

The construction of $\hat{G}$ discussed in Sections 2 and 3 depends on the choice of a $\Lambda \in L_1$ which generates the cyclic group L_1/L_0. Here we show that if we replace Λ by Λ', where $\Lambda' + L_0 = (\pm\Lambda) + L_0$, then the corresponding group $\hat{G}'$ is isomorphic to $\hat{G}$ in a way which is "compatible" with the diagrams (1) in which they lie.

First, suppose that $\Lambda' = -\Lambda$. For each mapping involved in the construction of $\hat{G}$, the corresponding mappings for $\hat{G}'$ will carry the same name with the addition of a prime; for example ψ', det$'$, etc. Define $\phi : \hat{H} \to \hat{H}$ by $\phi(f)(\alpha_j) = f(\alpha_j)$, $1 \leq j \leq l$, and $\phi(f)(\alpha_0) = f(\alpha_0)^{-1}$, for all $f \in H$. Then ϕ is an automorphism of $\hat{H}$ and $\sigma: G \times_\psi \hat{H} \to G \times_{\psi'} \hat{H}$ defined by $(g, \hat{h}) \mapsto (g, \phi(\hat{h}))$ is an isomorphism. We have $i' = \phi i$ and $\det(\hat{h}) = \hat{h}(w)$ for all $\hat{h} \in \hat{H}$, see Section 2. Similarly, we have $w' = -(z_0\alpha_0 - (z_1\alpha_1 + \cdots + z_l\alpha_l))$, from which it follows that $\det'(\phi(\hat{h})) = \phi(\hat{h})(w') = \det(\hat{h})^{-1}$. We obtain from this that $\delta(M) = M'$, and σ induces an isomorphism $\hat{\sigma}: \hat{G} \to \hat{G}'$. It is easy to see that if the diagrams (1) for $\hat{G}$ and $\hat{G}'$ are joined by means of the mapping $\hat{\sigma}$, the maps $x \to x^{-1}$ on all groups involving K^*, and identity maps everywhere else, the resulting diagram is commutative.

Now suppose $\Lambda' \equiv \Lambda (\mathrm{mod}\, L_0)$. We use the above conventions on notation. Note that for $n\Lambda = z_1\alpha_1 + \cdots + z_l\alpha_l$ and $n\Lambda' = z'_1\alpha_1 + \cdots + z'_l\alpha_l$, we have $w = -(z_0\alpha_0 + \cdots + z_l\alpha_l)$ and $w' = -(z_0\alpha_0 + z'_1\alpha_1 + \cdots + z'_l\alpha_l)$, with $z_0 = n$ and $w - w' = -n(\Lambda - \Lambda')$.

Define $\phi : \hat{H} \to \hat{H}$ by $\phi(f)(\alpha_j) = f(\alpha_j)$, $1 \leq j \leq l$, and $\phi(f)(\alpha_0) = f(\Lambda - \Lambda')f(\alpha_0)$. Then ϕ is an automorphism of $\hat{H}$ and $\delta: G \times_\psi \hat{H} \to G \times_{\psi'} \hat{H}$ defined by $(g, \hat{h}) \mapsto (g, \phi(\hat{h}))$ is an isomorphism. One checks that $i' = \phi i$ and $\det'\phi = \det$. Then $\delta(M) = M'$ and δ induces an isomorphism $\hat{\delta} : \hat{G} \to \hat{G}'$. This time the two corresponding diagrams (1) join together by $\hat{\delta}$ and identity maps everywhere else to give a commutative diagram.

It follows from these remarks that the construction of $\hat{G}$ is essentially unique in all cases except possibly A_l, and D_l when l is even. In fact, the reader may quite easily show that the arguments above can be extended to the case of D_l, l even, so that the construction is unique in this case too. The situation for type A_l is unknown.

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REFERENCES

1. N. Bourbaki, *Groupes et algèbres de Lie*, IV, V, VI, Herman, Paris, 1968.

2. R. W. Carter, *Simple Groups of Lie Type*, John Wiley and Sons, New York, 1972.

3. J. Dieudonné, *La Géométrie des groupes classiques*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, 5, Springer, 1955.

4. G. B. Seligman, *Modular Lie Algebras*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, 40, Springer, 1967.

5. R. Steinberg, *Lectures on Chevalley Groups*, Yale University, 1967.

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